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Unsteady heat transfer from a circular cylinder immersed in a Darcy flow

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SUMMARY

Analyses are made for unsteady heat transfer from a circular cylinder immersed in a porous medium through which a liquid is flowing according to Darcy's law. Asymptotic solutions for large and small Peclet numbers (Pe) are obtained for the case where the unsteady temperature field is produced by a step change in wall temperature. The former is valid for $Pe \gtrsim 200$ and the latter for $Pe \lesssim 0.1$. The series solution for small time, which is valid for all values of Pe, is also obtained. By applying the Euler transformation to the series, its convergence is greatly improved, and it appears that the Eulerized series determines the mean Nusselt number adequately for most values of time.

1. Introduction

Heat transfer from a surface immersed in a porous medium through which a liquid is flowing is of great practical importance in many branches of engineering. As is well known, the superficial velocity (volumetric flow rate per unit cross-section area) is governed by Darcy's law as long as the Reynolds number, Re_b, based on the averaged grain diameter is smaller than about 10 [1]. In the present paper, analytical solutions of the energy equation are obtained for unsteady heat transfer from a circular cylinder immersed in a Darcy flow. In order to simplify the problem, the following assumptions are made: (1) Every grain of the porous medium is so small that we can define continuous velocity and temperature fields by taking locally averaged mass flow and locally averaged temperature, respectively; (2) All the properties of the porous medium and the fluid are homogeneous and isotropic, so that the locally averaged velocity is given by a potential flow solution; (3) The difference in temperature between the fluid and the solid composing the porous medium is negligible; (4) The dispersive heat flux resulting from velocity and temperature fluctuations in the pore space is negligible compared with the conductive one.

In many cases of heat transfer in porous media, especially at low Reynolds number flows, the assumption (3) holds [1]. The assumption (4) is valid when the Peclet number Pe_b based on the average grain diameter is smaller than about 3000 [1].

Under the above assumptions, the energy equation for the locally averaged temperature is greatly simplified and becomes identical in dimensionless form with the ordinary one for a fluid continuum having a velocity distribution given by potential theory. For the steady-state case, many authors [2, 3, 4] have obtained analytical solutions of the energy equation for heat transfer from a circular cylinder immersed in a potential flow. These solutions can be used for heat transfer in a Darcy flow provided that the above assumptions are satisfied. For the unsteady case, however, only a few studies have been made on heat transfer in a potential flow. Recent contributions on unsteady heat transfer from a circular cylinder have been made by Okada et al. [5, 6], who carried out both numerical calculations and experiments for the case where the unsteady temperature field is produced by sudden imposition of a constant temperature difference between the cylinder and its surroundings. They showed that the agreement in temperature distribution between the calculated values and the experimental ones is satisfactory. Their investigation is limited to several values of *Pe* (Peclet number based on the radius of the cylinder) between about $1 \sim 50$.

The present paper reconsiders this problem theoretically, giving asymptotic solutions of the unsteady energy equation for large and small values of *Pe*. These solutions are valid for $Pe \gtrsim 200$ and $Pe \lesssim 0.1$, respectively. Furthermore, the series solution for small time, which is valid for all values of *Pe*, is also obtained. The convergence of this short-time solution becomes poor as time increases. It is, however, shown that if the Euler transformation is applied to the series its convergence greatly improves, and it gives satisfactory results for the mean Nusselt number for most values of time.

2. Governing equation

Consider a Darcy flow around a circular cylinder of radius r_0. The superficial velocity of the flow far upstream is assumed to be uniform (= U). Under the assumptions made in the preceding section, the energy equation can be written as

$$[n\rho_f c_f + (1-n)\rho_s c_s]\partial t'/\partial \tau' + \rho_f c_f \left(u'\partial t'/\partial r' + \frac{v'}{r'}\,\partial t'/\partial\theta\right)$$

$$= \lambda_c \nabla_{r'}^2 t', \tag{1}$$

where

$$\nabla_{r'}^2 = \partial^2/\partial r'^2 + \frac{1}{r'}\,\partial/\partial r' + \frac{1}{r'^2}\,\partial^2/\partial\theta^2, \tag{2}$$

and

$$u' = -U(1 - r_0^2/r'^2)\cos\theta,$$
$$v' = U(1 + r_0^2/r'^2)\sin\theta. \tag{3}$$

In the above equations, t' is the locally averaged temperature, τ' the time, (r', θ) polar co-ordinates with origin at the center of the cylinder, u' and v' the superficial velocities in r'- and θ-direction, respectively, ρ_f and c_f the density and specific heat of the fluid, ρ_s and c_s the density and specific heat of the solid composing the porous medium, λ_c the effective thermal conductivity of the saturated porous medium and n the volumetric porosity. We assume that, initially, the surface of the cylinder and the surroundings are at the same temperature T_∞, whereupon at time $\tau' = 0$ the surface temperature is suddenly changed to a constant value T_w. Then, the boundary conditions of the present problem can be written as

$$\begin{aligned} &\tau' < 0: \quad t' = T_\infty, \\ &\tau' \geqq 0: \quad t' = T_w \quad \text{at} \quad r' = r_0, \\ &\qquad\qquad t' \to T_\infty \quad \text{as} \quad r' \to \infty. \end{aligned} \tag{4}$$

Defining such dimensionless quantities as

$$\begin{aligned} &r = r'/r_0, \quad \tau = \frac{\rho_f c_f}{n\rho_f c_f + (1-n)\rho_s c_s} U\tau'/r_0, \\ &t = (t' - T_\infty)/(T_w - T_\infty), \quad Pe = \rho_f c_f U r_0/\lambda_c, \end{aligned} \tag{5}$$

(1) and (4) can be written as

$$\partial t/\partial\tau + \left(u\partial t/\partial r + \frac{v}{r}\,\partial t/\partial\theta\right) = Pe^{-1}\nabla_r^2 t, \tag{6}$$

where ∇_r^2 is a dimensionless form of $\nabla_{r'}^2$ and

$$u = -(1 - r^{-2})\cos\theta, \quad v = (1 + r^{-2})\sin\theta. \tag{7}$$

The dimensionless boundary conditions become

$$\begin{aligned} &\tau < 0: \quad t = 0, \\ &\tau \geqq 0: \quad t = 1 \quad \text{at} \quad r = 1, \\ &\qquad\qquad t \to 0 \quad \text{as} \quad r \to \infty, \end{aligned} \tag{8}$$

respectively.

3. Solutions for small Peclet numbers

In this section, we shall obtain asymptotic solutions of (6) for small Peclet numbers. In the limit $Pe \to 0$, the non-steady term $\partial t/\partial\tau$ should be balanced by the conduction term. In order to reflect this balance in the energy equation, we introduce new variables

$$\hat{\tau} = \tau/Pr, \quad \hat{t}(\hat{\tau},r,\theta) = t(\tau,r,\theta), \tag{9}$$

in terms of which (6) can be written as

$$\partial\hat{t}/\partial\hat{\tau} + Pe\left(u\partial\hat{t}/\partial r + \frac{v}{r}\,\partial\hat{t}/\partial\theta\right) = \nabla_r^2\hat{t}. \tag{10}$$

The solution of (10) is assumed to be of the form

$$\hat{t} = \hat{t}_0(\hat{\tau},r,\theta) + Pe\hat{t}_1(\hat{\tau},r,\theta) + Pe^2\hat{t}_2(\hat{\tau},r,\theta) + \ldots \tag{11}$$

Substituting this into (10), we have

$$\begin{aligned} &\partial\hat{t}_0/\partial\hat{\tau} - \nabla_r^2\hat{t}_0 = 0,\\ &\partial\hat{t}_n/\partial\hat{\tau} - \nabla_r^2\hat{t}_n = -\left(u\partial\hat{t}_{n-1}/\partial r + \frac{v}{r}\,\partial\hat{t}_{n-1}/\partial\theta\right) \quad \text{for } n \geqq 1. \end{aligned} \tag{12}$$

Boundary conditions for these equations are

$$\begin{aligned} &\hat{t}_0(0,r,\theta) = 0, \quad \hat{t}_0(\hat{\tau},1,\theta) = 1, \quad \hat{t}_0(\hat{\tau},\infty,\theta) = 0,\\ &\hat{t}_n(0,r,\theta) = \hat{t}_n(\hat{\tau},1,\theta) = \hat{t}_n(\hat{\tau},\infty,\theta) = 0 \quad \text{for } n \geqq 1. \end{aligned} \tag{13}$$

The required solution for $\hat{t}_0$ has already been obtained by Carslaw and Jaeger [7] as

$$\hat{t}_0 = 1 + \frac{2}{\pi}\int_0^\infty \frac{\exp(-u^2\hat{\tau})}{u}\,\frac{J_0(ur)Y_0(u) - Y_0(ur)J_0(u)}{J_0^2(u) + Y_0^2(u)}\,du, \tag{14}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions of first and second kind, respectively. Numerical values of $\hat{t}_0$ as a function of r for various values of time have been given by Jaeger [8].

It should be noted at this point that for $\hat{\tau} \to \infty$ (steady state) the asymptotic expression for the surface heat transfer for $Pe \to 0$ has already been given as [3]

$$-(\partial t/\partial r)_{r=1} = \Delta + \tfrac{1}{4}Pe^2(-2 + \Delta) + O(Pe^4), \tag{15}$$

where

$$\Delta = [\ln(4/Pe) - C]^{-1}, \tag{16}$$

$C = 0.5772156649...$ being Euler's constant. It is clear that the series solution (11) cannot approach this steady-state solution as $\hat{\tau}$ tends to infinity. This fact means that the equation (10) and the expansion (11) are valid only in the small-time region where $\hat{\tau} = O(1)$ ($\tau = O(Pe)$). The reason for the failure of the expansion (11) at large $\hat{\tau}$ can be explained as follows. In the small-time region, the thermal layer is restricted to the vicinity of the cylinder (inner region), where the effect of the convection term on the temperature is small, and therefore the expansion (11) is valid throughout the temperature field. As $\hat{\tau}$ becomes larger, however, the thermal layer expands into the outer region far from the cylinder, where, no matter how small Pe is, both the convection and the conduction terms are of the same order of magnitude. It is the existence of this outer region which prevents the expansion (11) from becoming a uniformly valid approximation to the function $\hat{t}$.

In order to obtain a uniformly valid solution, we shall employ the method of matched asymptotic expansions. From the argument stated above, it is reasonable to construct two expansions valid in the large-time region, in addition to (11), an 'inner' and an 'outer' expansion respectively. Following a well-established procedure of the method, we construct the inner and the outer expansions in such a way that: (1) The inner expansion, which is valid in the inner region near the surface, satisfies the boundary condition on the surface; (2) The outer expansion, which is valid in the outer region far from the surface, satisfies the boundary condition at infinity, (3) The two expansions match identically in the overlapping domain in space where both expansions are valid and also match the small-time solution (11) at small values of τ^*, τ^* being a time variable in the large-time region.

As the time variable in the large-time region, we introduce τ^* such as

$$\tau^* = Pe\tau = Pe^2\hat{\tau}, \tag{17}$$

in terms of which the energy equation can be written as

$$Pe^2 \partial t^*/\partial\tau^* + Pe(u\partial t^*/\partial r + \frac{v}{r}\partial t^*/\partial\theta) = \nabla_r^2 t^*, \tag{18}$$

where

$$t^*(\tau^*,r,\theta) = t(\tau,r,\theta). \tag{19}$$

The equation (18) is valid only in the inner region and shows that in this region thermal diffusion predominates as in the small-time region. The form of the asymptotic expansion for t^* may be determined from the asymptotic behaviour for $\hat{\tau} \to \infty$ of the small-time solution. From (11) and (14), the asymptotic behaviour of $(\partial\hat{t}/\partial r)_{r=1}$ for large value of $\hat{\tau}$ can be written as

$$-(\partial\hat{t}/\partial r)_{r=1} \sim 2\left[\frac{1}{\ln(4\hat{\tau}) - 2C} - \frac{C}{[\ln(4\hat{\tau}) - 2C]^2} + \frac{C^2 - \pi^2/6}{[\ln(4\hat{\tau}) - 2C]^3} + \ldots\right] + O(Pe). \tag{20}$$

From (20) and the requirement that the expansion for t^* should match the small-time solution, the asymptotic behaviour of $(\partial t^*/\partial r)_{r=1}$ for small τ^* should be of the form

$$-(\partial t^*/\partial r)_{r=1} \sim \Delta + \tfrac{1}{2}\Delta^2(-\ln\tau^* + \ln 4 - C) + \Delta^3\,[\tfrac{1}{4}(\ln 4 - \ln\tau^*)^2 - \frac{C}{2}(\ln 4 - \ln\tau^*) + \tfrac{1}{4}(C^2 - \pi^2/6)] + \ldots + O(Pe). \tag{21}$$

In view of this, the solution of (18) may be expanded as

$$t^* = t_0^*(\tau^*,r,\theta) + \Delta t_1^*(\tau^*,r,\theta) + \Delta^2 t_2^*(\tau^*,r,\theta) + \ldots + O(Pe). \tag{22}$$

Substituting this into (18), we have, for all n,

$$\nabla_r^2 t_n^* = 0. \tag{23}$$

It is seen that the unsteady term disappears in (23). This means that the t_n^*'s can be determined without imposing the matching condition (21) upon them. In constructing solutions, the matching condition between the large-time and the small-time solutions is imposed only in the outer field. It will be shown later that the inner solution obtained below automatically matches the small-time solution.

Application of the Laplace transformations

$$\overline{t^*} = \int_0^\infty t^* \exp(-p\tau^*) d\tau^* \quad \text{and} \quad \overline{t_n^*} = \int_0^\infty t_n^* \exp(-p\tau^*) d\tau^*, \tag{24}$$

in (22) and (23) yields

$$\overline{t^*} = \overline{t_0^*} + \Delta \overline{t_1^*} + \Delta^2 \overline{t_2^*} + \Delta^3 \overline{t_3^*} + \ldots + O(Pe), \tag{25}$$

and

$$\nabla_r^2 \overline{t_n^*} = 0. \tag{26}$$

The solutions of (26) satisfying the boundary condition on the surface, $\overline{t_0^*} = 1/p$ and $\overline{t_n^*} = 0$ ($n \geqq 1$) at $r = 1$, are

$$\overline{t_0^*} = 1/p$$

and (27)

$$\overline{t_n^*} = a_n \ln r \quad \text{for } n \geqq 1,$$

where the a_n's are integration constants to be determined from the matching condition between the inner and the outer solutions.

We shall next consider the outer region. In this region, we introduce the following outer variables

$$\rho = Per, \quad \overline{T^*}(\rho,\theta;p) = \overline{t^*}(r,\theta;p), \quad [T^*(\tau^*,\rho,\theta) = t^*(\tau^*,r,\theta)], \tag{28}$$

in terms of which the energy equation can be written as

$$p\overline{T^*} - T^*(0,\rho,\theta) - (1 - \rho^{-2}Pe^2)\cos\theta\,\partial\overline{T^*}/\partial\rho + (\rho^{-1} - \rho^{-3}Pe^2)\sin\theta\,\partial\overline{T^*}/\partial\theta$$
$$= \nabla_\rho^2 \overline{T^*}, \tag{29}$$

where ∇_ρ^2 is the same operator as ∇_r^2 but with r replaced by ρ. This equation reflects a proper balance between the convection and the conduction terms. The term $T^*(0, \rho, \theta)$ can be determined by applying the matching condition between T^* and $\hat{t}$. For any finite $\hat{\tau}$, the solution for $\hat{t}_0$, (14), vanishes exponentially as $r \to \infty$, meaning that in the small-time region the temperature rise is negligible in the outer field. Hence, the matching condition requires that the following equation holds:

$$T^*(0,\rho,\theta) = 0. \tag{30}$$

The form of the expansion for T^* may be determined from the following matching condition:

$$\lim_{\rho \to 0} \overline{T^*} = \lim_{r \to \infty} \overline{\bar{t}^*}, \tag{31}$$

as $Pe \to 0$. From (25) and (27), this matching condition may be written as

$$\overline{T^*} \sim 1/p + a_1 + [a_1 \ln\rho + a_2 - (\ln 4 - C)a_1]\Delta + [a_2 \ln\rho + a_3 - (\ln 4 - C)a_2]\Delta^2$$
$$+ \ldots + O(Pe). \tag{32}$$

for small ρ. In view of this, we can assume that the solution of (29) may be expanded as

$$\overline{T^*} = \overline{T_0^*}(\rho,\theta;p) + \Delta\overline{T_1^*}(\rho,\theta;p) + \Delta^2\overline{T_2^*}(\rho,\theta;p) + \Delta^3\overline{T_3^*}(\rho,\theta;p)$$
$$+ \ldots + O(Pe). \tag{33}$$

Substitution of (33) into (29) gives the following equation for $\overline{T_n^*}$:

$$p\overline{T_n^*} - \cos\theta\,\partial\overline{T_n^*}/\partial\rho + (\sin\theta/\rho)\partial\overline{T_n^*}/\partial\theta = \nabla_\rho^2\overline{T_n^*}. \tag{34}$$

The solution of this equation satisfying the boundary condition at infinity can easily be obtained as

$$\overline{T_n^*} = A_n K_0(\sqrt{1+4p}\,\rho/2)\exp(\rho\mu/2), \tag{35}$$

where the A_n's are integration constants, $K_0(z)$ denotes the modified Bessel function and

$$\mu = -\cos\theta. \tag{36}$$

From (33) and (35), the asymptotic behaviour of $\overline{T^*}$ for small ρ can be written as

$$\overline{T^*} \sim (A_0 + A_1\Delta + A_2\Delta^2 + A_3\Delta^3 + \ldots)(-\ln\rho + \ln 4 - \ln(1+4p)^{\frac{1}{2}} - C + O(\rho))$$
$$+ O(Pe). \tag{37}$$

Comparison of (32) and (37) determines the unknown constants a_n and A_n as follows:

$$\left.\begin{aligned} A_0 &= 0, \\ A_n &= (\ln\sqrt{1+4p})^{n-1}/p, \\ a_n &= -(\ln\sqrt{1+4p})^{n-1}/p \end{aligned}\right\} \text{ for } n \geq 1. \tag{38}$$

Thus, $\overline{t_n^*}$ and $\overline{T_n^*}$ are determined completely. By taking the inverse of $\overline{t_n^*}$ with the help of the tables in [9], we have

$$t^* = 1 - \Delta\ln r - \tfrac{1}{2}\Delta^2 E_1(\tau^*/4)\ln r - \tfrac{1}{4}\Delta^3 \Bigg[\int_0^1 E_1(\tau^* v/4)E_1(\tau^*(1-v)/4)dv$$
$$- (8/\tau^*)\left\{\left(\ln\frac{\tau^*}{4} + C\right)\exp(-\tau^*/4) + E_1(\tau^*/4)\right\}\Bigg]\ln r + O(\Delta^4) + O(Pe), \tag{39}$$

where

$$E_1(x) = \int_x^\infty \frac{\exp(-t)}{t}\, dt. \tag{40}$$

It is impossible to express T^* in terms of tabulated functions.

From (11) and (14), the wall temperature gradient may be obtained as

$$-(\partial\hat{t}/\partial r)_{r=1} = \frac{4}{\pi^2}\int_0^\infty \frac{\exp(-u^2\hat{\tau})du}{u[J_0^2(u) + Y_0^2(u)]} + O(Pe), \tag{41}$$

in the small-time region and, from (39), as

$$-(\partial t^*/\partial r)_{r=1} = \Delta + \tfrac{1}{2}\Delta^2 E_1(\tau^*/4) + \tfrac{1}{4}\Delta^3 \Bigg[\int_0^1 E_1(\tau^* v/4)E_1(\tau^*(1-v)/4)dv$$
$$- (8/\tau^*)\{(\ln(\tau^*/4) + C)\exp(-\tau^*/4) + E_1(\tau^*/4)\}\Bigg] + O(\Delta^4) + O(Pe), \tag{42}$$

in the large-time region. Numerical values of the integral in (41) have been tabulated by Jaeger and Clarke [10]. It is easy to verify that the asymptotic behaviour of $(\partial t^*/\partial r)_{r=1}$ for small τ^* calculated from (42) is in complete agreement with (21). Thus, the inner solution automatically matches the small-time solution.

From (41) and (42), a single composite expansion for the local Nusselt number $Nu = -(\partial t/\partial r)_{r=1}$ which is uniformly valid for all values of time can be constructed by adding (41) and (42) and then subtracting the common part. The result is

$$Nu = -(\partial t/\partial r)_{r=1} = \frac{4}{\pi^2}\int_0^\infty \frac{\exp(-u^2\hat{\tau})}{u\{J_0^2(u) + Y_0^2(u)\}}\, du + \tfrac{1}{2}\Delta^2\left\{E_1\left(\frac{\tau^*}{4}\right) + \ln\frac{\tau^*}{4} + C\right\}$$

$$+\Delta^3\left[\frac{1}{4}\int_0^1 E_1(\tau^*\nu/4)E_1(\tau^*(1-\nu)/4)d\nu\right.$$

$$-\frac{2}{\tau^*}\left\{\left(\ln\frac{\tau^*}{4}+C\right)\exp\left(-\frac{\tau^*}{4}\right)+E_1\left(\frac{\tau^*}{4}\right)\right\}$$

$$\left.-\frac{1}{4}\left(\ln\frac{\tau^*}{4}\right)^2-\frac{C}{2}\ln\frac{\tau^*}{4}-\frac{C^2}{4}+\frac{\pi^2}{24}\right]+O(\Delta^4)+O(Pe). \tag{43}$$

For $\tau\to\infty$, the present expansion gives $Nu = 0.321$, 0.263 and 0.185 for $Pe = 0.1$, 0.05 and 0.01, respectively, and these values are about 1.32%, 0.42% and 0.00% above the exact values $Nu = 0.3172$, 0.2617 and 0.1847. The convergence of the expansion (43) is found to be better for smaller values of time and therefore we can expect that the error of the present results for $\tau<\infty$ is smaller than that for $\tau\to\infty$. In Fig. 1, the timewise variations of the Nusselt number calculated from (43) are shown graphically for $Pe = 0.1$, 0.05 and 0.01.

Finally, it should be noted that in the analysis presented above we have used, as the velocity field in the energy equation, only the leading term in (7), namely, uniform stream. Therefore the results obtained in this section can be applied not only to Darcy flow, but also to all the problems in which the so-called 'Oseen approximation' is valid in the energy equation. It is clear that the second term in (7) will influence the temperature field when the analysis is continued up to the term of $O(Pe)$.

4. Solution for large Peclet number

We shall now proceed to obtain an asymptotic solution of (6) for large Peclet numbers. By introducing the following variable

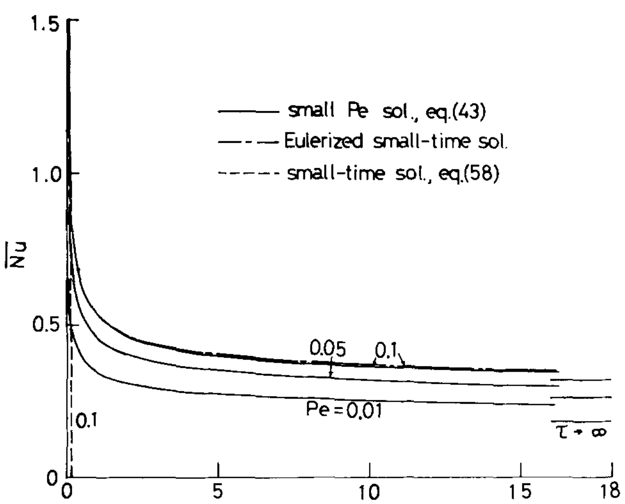

Fig. 1. Timewise variation in the mean Nusselt number for small Peclet numbers.

$$Y = Pe^{\frac{1}{2}}(r - 1). \tag{44}$$

we can write the energy equation (6) as

$$\begin{aligned} &\partial t/\partial \tau + (-2Y + 3Y^2 Pe^{-\frac{1}{2}} + \ldots)\cos\theta\,\partial t/\partial Y \\ &+ (2 - 2YPe^{-\frac{1}{2}} + \ldots)(1 - YPe^{-\frac{1}{2}} + \ldots)\sin\theta\,\partial t/\partial\theta \\ &= \partial^2 t/\partial Y^2 + Pe^{-\frac{1}{2}}\partial t/\partial Y + \ldots \end{aligned} \tag{45}$$

In the limit $Pe \to \infty$, (45) becomes

$$\partial t/\partial \tau - 2Y\cos\theta\,\partial t/\partial Y + 2\sin\theta\,\partial t/\partial\theta = \partial^2 t/\partial Y^2, \tag{46}$$

with

$$\begin{aligned} t &= 1 \quad \text{at} \quad Y = 0, \\ t &\to 0 \quad \text{as} \quad Y \to \infty. \end{aligned} \tag{47}$$

The solution of this equation can be obtained by using a method similar to that used by Ruckenstein [11] and Chao [12] for analyzing transient heat/mass transfer from a translating droplet. The final result for the present problem is easily found to be

$$t = \operatorname{erfc}(Y\sin\theta/\sqrt{2\zeta}), \tag{48}$$

where

$$\zeta = -\cos\theta + \left[1 - \frac{1-\cos\theta}{1+\cos\theta}\exp(-4\tau)\right] \Big/ \left[1 + \frac{1-\cos\theta}{1+\cos\theta}\exp(-4\tau)\right]. \tag{49}$$

From (48), the local Nusselt number can be expressed as

$$Nu = (2Pe/\pi\zeta)^{\frac{1}{2}}\sin\theta, \tag{50}$$

and the mean Nusselt number averaged over the surface as

$$\overline{Nu} \equiv \frac{1}{\pi}\int_0^\pi Nu\,d\theta = \frac{4}{\pi}\sqrt{\frac{Pe}{\pi(1-\exp(-4\tau))}}\,E(\sqrt{1-\exp(-4\tau)}), \tag{51}$$

where $E(k)$ is the complete elliptic integral of the second kind. The local Nusselt number distributions around the cylinder calculated from (50) are shown graphically in Fig. 2 for various values of time. It is seen that the steady state is almost reached at $\tau = 1$. In Fig. 3, the timewise variation of the mean Nusselt number calculated from (51) is shown by a solid line.

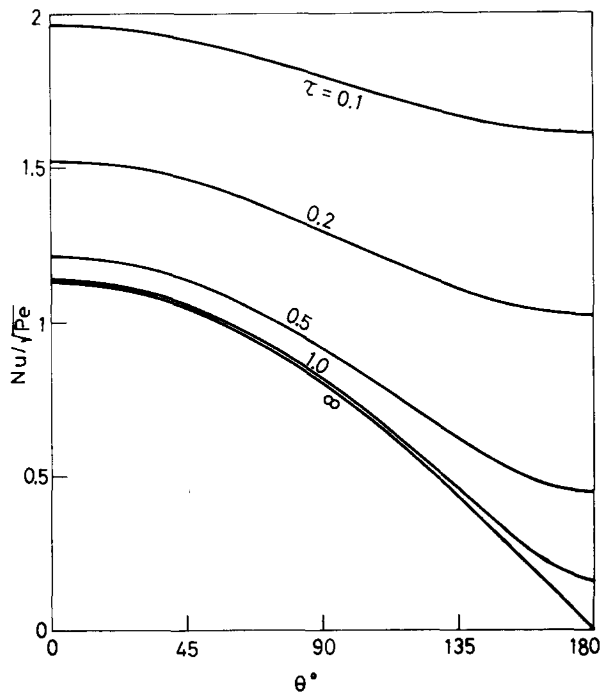

Fig. 3. Timewise variation in the mean Nusselt number for $Pe \to \infty$.

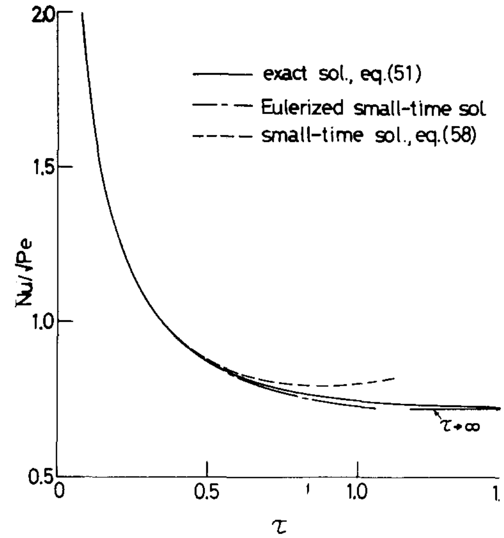

Fig. 2. Local Nusselt number distributions around a cylinder for $Pe \to \infty$.

5. Solution for small times

The fact that the heat transfer process when τ is small is dominated mainly by diffusion from the surface of the cylinder suggests the introduction of the following new variable

$$\eta = \tfrac{1}{2}(Pe/\tau)^{\frac{1}{2}}(r-1) = \tfrac{1}{2}Y\tau^{-\frac{1}{2}}. \tag{52}$$

Furthermore, we assume the solution of (6) to be of the form

$$t = t_0(\eta,\theta) + \tau^{\frac{1}{2}}t_1(\eta,\theta) + \tau t_2(\eta,\theta) + \tau^{\frac{3}{2}}t_3(\eta,\theta) + \ldots \tag{53}$$

Substituting (52) and (53) into (6) and expanding all terms for small τ, we have the following set of the equations for the t_n's:

$$\begin{aligned} &t_0'' + 2\eta t_0' = 0,\\ &t_1'' + 2\eta t_1' - 2t_1 = 4\exp(-\eta^2)/(\pi Pe)^{\frac{1}{2}},\\ &t_2'' + 2\eta t_2' - 4t_2 = (4/\sqrt{\pi})(2\cos\theta - 3/Pe)\eta\exp(-\eta^2) + (2/Pe)\mathrm{erfc}\eta, \quad \ldots, \end{aligned} \tag{54}$$

where primes denote differentiation with respect to η, with boundary conditions

$$t_0 = 1 \quad \text{at} \quad \eta = 0, \qquad t_0 \to 0 \quad \text{as} \quad \eta \to \infty,$$
$$t_n = 0 \quad \text{at} \quad \eta = 0, \qquad t_n \to 0 \quad \text{as} \quad \eta \to \infty \quad (\text{for } n \geqq 1). \tag{55}$$

These equations can be solved in a straightforward manner. The solutions are obtained for t_0, $t_1, \ldots, t_6$, among which only t_0, t_1 and t_2 are shown below:

$$\begin{aligned} t_0 &= \operatorname{erfc}\eta, \\ t_1 &= -Pe^{-\frac{1}{2}}\eta \operatorname{erfc}\eta, \\ t_2 &= Pe^{-1}[\eta^2 \operatorname{erfc}\eta + (1/2\sqrt{\pi})\eta \exp(-\eta^2)] - (2\cos\theta/\sqrt{\pi})\eta\exp(-\eta^2), \quad \ldots . \end{aligned} \tag{56}$$

From these solutions we can obtain the following expansion for the local Nusselt number:

$$\begin{aligned} Nu/\sqrt{Pe} &= (\pi\tau)^{-\frac{1}{2}} + \tfrac{1}{2}Pe^{-\frac{1}{2}} + (\tau/\pi)^{\frac{1}{2}}\left(-\frac{1}{4Pe} + \cos\theta\right) \\ &+ \tfrac{1}{2}Pe^{-\frac{1}{2}}\tau\left(\frac{1}{4Pe} - \cos\theta\right) \\ &+ \pi^{-\frac{1}{2}}\tau^{\frac{3}{2}}\left(-\frac{25}{96Pe^2} + \frac{11\cos\theta}{12Pe} - \frac{\cos^2\theta}{2} + \frac{2}{3}\right) \\ &+ \tfrac{1}{4}Pe^{-\frac{1}{2}}\tau^2\left(\frac{13}{16Pe^2} - \frac{5\cos\theta}{2Pe} + 3\cos^2\theta - \frac{17}{4}\right) \\ &+ \pi^{-\frac{1}{2}}\tau^{\frac{5}{2}}\left(-\frac{1073}{1920Pe^3} + \frac{743\cos\theta}{480Pe^2} - \frac{19\cos^2\theta}{8Pe} + \frac{21}{5Pe}\right. \\ &\left. + \frac{\cos^3\theta}{2} - \frac{2\cos\theta}{3}\right) + \ldots, \end{aligned} \tag{57}$$

and, from this, the mean Nusselt number may be obtained as

$$\begin{aligned} \overline{Nu}/\sqrt{Pe} &= (\pi\tau)^{-\frac{1}{2}} + \tfrac{1}{2}Pe^{-\frac{1}{2}} - (\tau/\pi)^{\frac{1}{2}}/4Pe + \tfrac{1}{8}Pe^{-\frac{3}{2}}\tau \\ &+ \pi^{-\frac{1}{2}}\tau^{\frac{3}{2}}\left(-\frac{25}{96Pe^2} + \frac{5}{12}\right) + Pe^{-\frac{1}{2}}\tau^2\left(\frac{13}{64Pe^2} - \frac{11}{16}\right) \\ &+ \pi^{-\frac{1}{2}}\tau^{\frac{5}{2}}\left(-\frac{1073}{1920Pe^3} + \frac{241}{80Pe}\right) + \ldots \end{aligned} \tag{58}$$

These expansions are valid for all values of Pe, but only for small τ. It is possible, however, to improve their convergence by recasting them in powers of the new parameter defined by (Euler transformation)

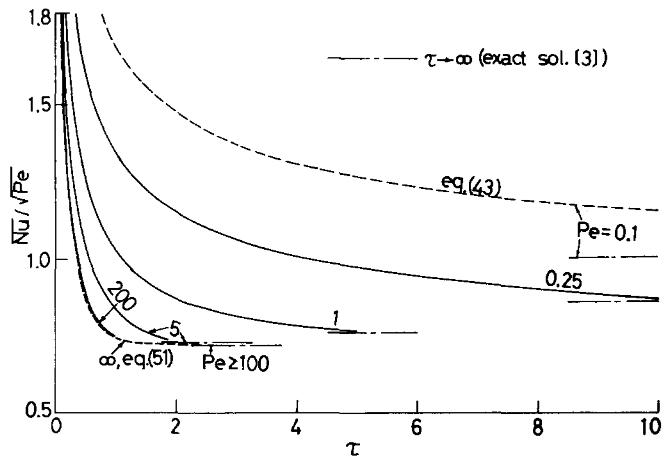

Fig. 4. Timewise variation in the mean Nusselt number calculated from the Eulerized small-time solution.

$$\epsilon = \tau^{\frac{1}{2}}/(1 + \tau^{\frac{1}{2}}), \tag{59}$$

or, when Pe is very large, by

$$\epsilon = \tau^2/(1 + \tau^2). \tag{60}$$

The mean Nusselt number calculated from (58) and that from the Eulerized series are shown for $Pe = 0.01$ and ∞ in Figs. 1 and 3, respectively. It is seen that a remarkable improvement can be achieved by the Eulerized series and it appears that the Eulerized series predicts sufficiently reliable values of the mean Nusselt number for most values of time. In Fig. 4 we show the mean Nusselt number results calculated from the Eulerized series for $0.1 < Pe < \infty$. It is seen that as Pe decreases a larger value of τ is required to approach its steady-state value. The difference in the value of $\overline{Nu}/\sqrt{Pe}$ between $Pe = 200$ and ∞ is within about 3%, so that for practical purposes we can use the solution for $Pe = \infty$ to calculate the mean Nusselt number for $Pe \gtrsim 200$.

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